METRIC SPACES

In this section, we generalize the notion of 'distance' to topological spaces.

DEFINITION: A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ which satisfies:

- $\forall a, b \in X,\ d(a,b) \geq 0$ and $d(a,b) = 0$ iff $a = b$.
- $\forall a, b \in X,\ d(a,b) = d(b,a)$.
- **Triangle Inequality**: $\forall a, b, c \in X,\ d(a,b) \leq d(a,c) + d(c,b)$

EXAMPLES:

- For $X = \mathbb{R}$, $d(x_0, x_1) = |x_1 - x_0|$ is a metric. (This is the **Euclidean Metric**.)
- For $X = \mathbb{R}^2$, the following are metrics:
  - **Euclidean Metric:** $d((x_0, y_0), (x_1, y_1)) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$
  - **Taxicab Metric:** $T((x_0, y_0), (x_1, y_1)) = |x_1 - x_0| + |y_1 - y_0|$
  - **Max Metric**: $M((x_0, y_0), (x_1, y_1)) = \max\{|x_1 - x_0|, |y_1 - y_0|\}$

  NOTE: All three of the given metrics for $\mathbb{R}^2$ reduce to the Euclidean Metric on $\mathbb{R}$ when $y_0 = y_1 = 0$.
- Given $X \neq \emptyset$, the **Discrete Metric** is defined as follows: $\delta(a,b) = 0$ if $a = b$ and $\delta(a,b) = 1$ otherwise.

DEFINITIONS:

- A **metric space** is a pair (X, d) where X is a nonempty set and d is a metric on X.
- For $x_0 \in X$, the set $N_d(x_0, \epsilon) = \{x \in X : d(x, x_0) < \epsilon\}$ is the **<u>open</u> ball centered at x_0 with radius ϵ.**
- For $x_0 \in X$, the set $B_d(x_0, \epsilon) = \{x \in X : d(x, x_0) \leq \epsilon\}$ is the **<u>closed</u> ball centered at x_0 with radius ϵ.**
- For $x_0 \in X$, the set $S_d(x_0, \epsilon) = \{x \in X : d(x, x_0) = \epsilon\}$ is the **sphere centered at x_0 with radius ϵ.**
- The **topology generated by** d, $\mathcal{T}_d$ is the topology with base $\{N_d(x_0, \epsilon) : x_0 \in X,\ \epsilon > 0\}$. That is:

$$T \in \mathcal{T}_d \iff \forall x_0 \in T\ \exists \epsilon > 0 \text{ such that } N_d(x_0, \epsilon) \subseteq T$$

EXAMPLE: Let $X = \mathbb{R}^2$. Sketch $S_d((0,0), 1)$ for each of the Euclidean, Taxicab, and Max Metrics.

EXAMPLE: Suppose $(X, \mathcal{T}_d)$ is a space with topology generated by the metric d.

- Show $N_d(x_0, \epsilon)$, the open ball centered at x_0 of radius ϵ is, indeed, open.
- Show $B_d(x_0, \epsilon)$, the closed ball centered at x_0 of radius ϵ is, indeed, closed.
- Show $\overline{N_d(x_0, \epsilon)} \subseteq B_d(x_0, \epsilon)$.
- Find an example where $\overline{N_d(x_0, \epsilon)} \neq B_d(x_0, \epsilon)$

EXAMPLE: Show all metric topologies are Hausdorff.

THEOREM: Suppose $F : (X, d) \to (Y, \rho)$ is a function from one metric space to another. Then:

$$F \text{ is continuous} \iff \forall\, x_0 \in X,\ \forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ such that } d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \epsilon$$

DEFINITION: Given a set X, two metrics d_1 and d_2 are called **equivalent** if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.

EXAMPLE: Two metrics d_1 and d_2 on X are equivalent iff $\forall\, x \in X$ and $r > 0$, $\exists\, r', r'' > 0$ such that:

$$N_{d_1}(x, r') \subseteq N_{d_2}(x, r) \quad \text{and} \quad N_{d_2}(x, r'') \subseteq N_{d_1}(x, r)$$

EXAMPLE: On $\mathbb{R}^2$, the Euclidean, Taxicab, and Max metrics are equivalent.

EXAMPLE: A metric d on a space X is said to be **bounded** iff $\exists\, M > 0$ such that $\forall\, a, b \in X$, $d(a, b) \leq M$.

EXAMPLE: The discrete metric is bounded.

EXAMPLE: Given a metric d on a set X with $a, b \in X$, define:

$$m(a, b) = \min\{d(a, b), 1\} \quad \text{and} \quad \rho(a, b) = \frac{d(a, b)}{1 + d(a, b)}$$

- Show m and ρ are metrics.
- Show m and ρ are equivalent to d.

HINTS: When proving the triangle inequality for ρ, note that $f(x) = \dfrac{x}{x+1}$ is increasing (since $f''(x) > 0$).
For equivalence, note that $d(x, y) = \dfrac{\rho(x, y)}{1 - \rho(x, y)}$

EXAMPLE: Let $\mathcal{C}[0, 1]$ be the set of continuous functions on $[0, 1]$ (using the Euclidean topology.)

Show the following are metrics on $\mathcal{C}[0, 1]$:

- $d(f, g) = \displaystyle\int_0^1 |f(x) - g(x)|\, dx$
- $L(f, g) = \sqrt{\displaystyle\int_0^1 [f(x) - g(x)]^2\, dx}$
- $M(f, g) = \text{maximum}\{|f(x) - g(x)| \,:\, 0 \leq x \leq 1\}$

Let $f(x) = x^2$ and $g(x) = x^3$. Find $d(x^2, x^3)$, $L(x^2, x^3)$ and $M(x^2, x^3)$.

EXAMPLE: Suppose (X, d) and (Y, ρ) are metric spaces with corresponding topologies $\mathcal{T}_d$ and $\mathcal{T}_\rho$.

- How would you define a metric m on $X \times Y$? Can you prove m is a metric?
- Does your metric generate the standard product topology $\mathcal{T}_d \times \mathcal{T}_\rho$